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Integral Equations for Electromagnetic Scattering by Perfect Conductors With Two-Dimensional Geometry

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In this paper, we derive various integral equations related to the scattering of time-harmonic electromagnetic fields by perfect conductors with 2-dimensional geometry. The fields may be expressed in terms of two solutions of a scalar wave eqaation and decomposed into E waves and H waves. We consider the case in which part, or all, of each of the perfectly conducting cylindrical scatterers may be infinitesimally thin, and show that a standard integral equation, used in the case of H waves, does not determine the current density on the infinitesimally thin parts of the scatterers. We derive an alternate integral equation which does not suffer from this defect. This equation has been used by J. L. Blue in the numerical solution of the problem of scattering by an infinitesimally thin strip.

I. INTRODUCTION

In this paper, we consider the scattering of time-harmonic electromagnetic fields by perfect conductors with 2-dimensional geometry, in which the boundaries are independent of the z-coordinate. The z-dependence of the fields is assumed to be of the form $\exp(ik \sin \alpha\, z)$, where k is the free space wave number and $|\alpha| < \pi/2$, so that the scattering of obliquely incident plane waves may be investigated. It is known[1] that the electromagnetic fields may be expressed in terms of the longitudinal components, E_z and H_z, and that each of these two quantities satisfies the scalar wave equation, with wave number $k \cos \alpha$. Moreover, since the boundary conditions on a perfectly conducting surface imply that both E_z and the normal derivative of H_z are zero, there is no coupling between E_z and H_z, and we refer to E waves and H waves, respectively.

Integral equations for scattering problems have been considered by numerous authors. A relatively recent treatment of this topic is that of

Poggio and Miller,[2] hut they give only a hrief discussion of the 2-dimensional case. A useful discussion of integral equations for the scalar prohlem is given hy Nohle.[3] Poggio and Miller state that the integral equation which they derive for the surface current in the case of H waves is useless when the scatterer is infinitely thin. Nohle points out that the corresponding integral equation, when applied to the problem of scattering hy an elliptic cylinder, degenerates as the eccentricity tends to unity, so that the scatterer hecomes an infinitesimally thin strip.

In this paper, we consider the case in which part, or all, of each of the perfectly conducting cylindrical scatterers may he infinitely thin. We derive an alternate integral equation for the current density on the scatterers, in the case of H waves, which does not degenerate on the infinitesimally thin segments. We discuss the relationship hetween this integral equation and one derived by Mitzner.[4] Our integral equation has been used hy Blue[5] in the numerical solution of the problem of scattering hy an infinitesimally thin strip. We also point out how the integral equation which does degenerate may he used to calculate H_z on both sides of the infinitesimally thin segments, once the entire current density on the scatterers is known. An integral equation which degenerates in the case of E waves is also derived, and this may be used analogously to calculate the values of the normal derivatives of E_z on hoth sides of the infinitesimally thin segments, once the entire current density on the scatterers is known.

In Section II we hriefly derive expressions for the transverse components of the field in terms of the longitudinal components E_z and H_z, and show that the latter quantities hoth satisfy a scalar wave equation. We also derive the houndary conditions on a perfectly conducting surface, and give an expression for the current density on the surface. The total fields are expressed as the sum of the incident and scattered fields. In Section III we derive an integral representation for the scattered field in terms of the total field, and its normal derivative, on the scatterers. A nondegenerate integral equation for the current density on the scatterers is ohtained in the case of E waves, by using this representation as a point on the houndary is approached. In the case of H waves, it is shown that the corresponding integral equation degenerates on the infinitesimally thin segments, since it contains an unknown quantity hesides the current density.

In Section IV we derive representations for the transverse component of the gradient of the scattered field. By using this representation to calculate the normal derivative of the scattered field as a point on the houndary is approached, we ohtain a nondegenerate integral equation for the current density on the scatterers in the case of H waves. In the case of E waves, it is shown that the corresponding

integral equation degenerates on the infinitesimally thin segments. The implications of these results are discussed.

II. THE ELECTROMAGNETIC FIELDS

We first write down equations which describe the electromagnetic fields due to scattering by perfect conductors with 2-dimensional geometry. If we suppress the factor $\exp(-i\omega t)$, where ω is the angular frequency, the divergenceless electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$, in free space, satisfy Maxwell's equations[6]

$$\nabla \times \mathbf{E} = i\omega\mu_0\mathbf{H}, \quad \nabla \times \mathbf{H} = -i\omega\epsilon_0\mathbf{E}, \tag{1}$$

where μ_0 is the permeability and ϵ_0 is the dielectric constant. The free space wave number is $k = \omega(\mu_0\epsilon_0)^{1/2}$. We consider the case of a 2-dimensional geometry in which the boundaries are independent of the coordinate z, and assume that the z-dependence of the fields is of the form $\exp(ik \sin \alpha\, z)$, where $|\alpha| < \pi/2$. This will allow us to consider scattering of obliquely incident plane waves. Accordingly, we now suppress the factor $\exp(ik \sin \alpha\, z)$, and write

$$\nabla = \nabla_t + ik \sin \alpha\, \mathbf{i}_z, \quad \mathbf{E} = \mathbf{E}_t + E_z\mathbf{i}_z, \quad \mathbf{H} = \mathbf{H}_t + H_z\mathbf{i}_z, \tag{2}$$

where $\mathbf{i}_z$ is a unit vector in the z-direction, and the subscript t refers to the transverse components.

If we split eqs. (1) into their transverse and longitudinal components, we obtain

$$\nabla_t E_z \times \mathbf{i}_z = ik \sin \alpha\, \mathbf{E}_t \times \mathbf{i}_z + i\omega\mu_0\mathbf{H}_t, \tag{3}$$

$$\nabla_t H_z \times \mathbf{i}_z = ik \sin \alpha\, \mathbf{H}_t \times \mathbf{i}_z - i\omega\epsilon_0\mathbf{E}_t, \tag{4}$$

and

$$\nabla_t \times \mathbf{E}_t = i\omega\mu_0 H_z\mathbf{i}_z, \quad \nabla_t \times \mathbf{H}_t = -i\omega\epsilon_0 E_z\mathbf{i}_z. \tag{5}$$

It is convenient to define the transverse wave number $k_t = k \cos \alpha$. Then, from (3) and (4), it follows that

$$k_t^2\mathbf{E}_t = i(k \sin \alpha\, \nabla_t E_z + \omega\mu_0\nabla_t H_z \times \mathbf{i}_z) \tag{6}$$

and

$$k_t^2\mathbf{H}_t = i(k \sin \alpha\, \nabla_t H_z - \omega\epsilon_0\nabla_t E_z \times \mathbf{i}_z). \tag{7}$$

Hence the transverse fields are expressed in terms of the longitudinal components. If we substitute these expressions for $\mathbf{E}_t$ and $\mathbf{H}_t$ into (5), we obtain

$$(\nabla_t^2 + k_t^2)E_z = 0, \quad (\nabla_t^2 + k_t^2)H_z = 0, \tag{8}$$

where we have used the relationships[7]

$$\nabla_t \times (\nabla_t V) = 0, \quad \nabla_t \times (\nabla_t V \times \mathbf{i}_z) = -\nabla_t^2 V\mathbf{i}_z. \tag{9}$$

Hence, as is known,[1] the longitudinal components of the field satisfy the scalar reduced wave equation.

The boundary conditions[8] on a perfectly conducting surface are that the tangential components of the electric field and the normal component of the magnetic field vanish, i.e.,

$$\mathbf{E} \times \mathbf{n} = 0, \quad \mathbf{H} \cdot \mathbf{n} = 0, \tag{10}$$

where $\mathbf{n}$ is a unit vector normal to the surface, directed into the scattering region. Because of the 2-dimensional geometry, $\mathbf{n} \cdot \mathbf{i}_z = 0$. From (2), (6) and (7), it follows that the boundary conditions (10) are equivalent to

$$E_z = 0, \quad \frac{\partial H_z}{\partial n} = 0 \quad \text{on a perfectly conducting surface.} \tag{11}$$

The current density on the surface[8] is $\mathbf{K} = \mathbf{n} \times \mathbf{H}$. We assume that t, $\mathbf{n}$ and $\mathbf{i}_z$ form a right-handed set of unit vectors. Then, from (2) and (7) we find that

$$\mathbf{K} = H_z \mathbf{t} + \frac{i}{k_t^2} \left(\omega \epsilon_0 \frac{\partial E_z}{\partial n} - k \sin \alpha \frac{\partial H_z}{\partial s} \right) \mathbf{i}_z, \tag{12}$$

where s denotes arc length along the cross-sectional boundary curve, and $\mathbf{t}$ is a unit vector tangent to the curve.

We write the total fields as the sum of incident and scattered fields,

$$\mathbf{E} = \mathbf{E}^i + \mathbf{E}^s, \quad \mathbf{H} = \mathbf{H}^i + \mathbf{H}^s. \tag{13}$$

In the case of incident plane waves we have, in Cartesian coordinates (x, y, z),

$$E_z^i = E_0 \exp[ik_t(x \cos \beta + y \sin \beta)], \quad H_z^i = 0, \tag{14}$$

for an E wave, and

$$H_z^i = H_0 \exp[ik_t(x \cos \beta + y \sin \beta)], \quad E_z^i = 0, \tag{15}$$

for an H wave. The factor $\exp[i(k \sin \alpha\, z - \omega t)]$ has been suppressed.

III. INTEGRAL EQUATIONS DERIVED FROM REPRESENTATION FOR THE SCATTERED FIELD

We first derive some integral representations for the scattered fields. We suppose that

$$(\nabla_t^2 + k_t^2)\psi^i = 0, \quad (\nabla_t^2 + k_t^2)\psi^s = 0, \tag{16}$$

and set $\psi^i = E_z^i$ and $\psi^s = E_z^s$, or $\psi^i = H_z^i$ and $\psi^s = H_z^s$, corresponding to E waves, or H waves, respectively. We let

$$\psi = \psi^i + \psi^s, \tag{17}$$

and then the corresponding boundary conditions are, from (11),

$$\psi = 0, \quad \text{or} \quad \frac{\partial \psi}{\partial n} = 0, \quad \text{on a perfectly conducting boundary.} \tag{18}$$

We introduce the 2-dimensional Green's function[3]

$$G(\mathbf{r}, \rho) = \frac{i}{4} H_0^{(1)}(k_t R), \tag{19}$$

where $H_0^{(1)}$ denotes a Hankel function[9] of zero order, and

$$\mathbf{r} = x\mathbf{i}_x + y\mathbf{i}_y, \quad \rho = \xi \mathbf{i}_x + \eta \mathbf{i}_y, \quad \mathbf{R} = \mathbf{r} - \rho, \quad R = |\mathbf{R}|, \tag{20}$$

where $\mathbf{i}_x$ and $\mathbf{i}_y$ are unit vectors in the x- and y-directions. Then[3]

$$(\nabla_t^2 + k_t^2) G = 0, \quad \mathbf{R} \neq 0, \tag{21}$$

We consider the case in which part, or all, of each perfectly conducting cylinder may be infinitesimally thin, and let the cross-sectional boundary curve of the jth cylinder be denoted by $C_j = \Gamma_j \cup L_j^+ \cup L_j^-$, where L_j^+ and L_j^- denote opposite sides of the infinitesimally thin segment(s) L_j. The segments L_j may be disjoint, as may be Γ_j also, as depicted in Fig. 1. The curves C_j are assumed to be piecewise differentiable. We consider a point $\mathbf{r}$ exterior to all the curves C_j, and apply Green's theorem[10] in the region A exterior to the curves C_j, exterior to $|\rho - \mathbf{r}| = \epsilon$, and interior to $|\rho| = \tau$, as depicted in Fig. 1. Then, with $C = \cup_j C_j$,

$$- \int_{C \cup |\rho - \mathbf{r}| = \epsilon \cup |\rho| = \tau} \left[\psi^s(\rho) \frac{\partial G}{\partial n} - G \frac{\partial \psi^s}{\partial n}(\rho) \right] ds = \int_A (\psi^s \nabla_t^2 G - G \nabla_t^2 \psi^s) \, dA = 0, \tag{22}$$

from (16) and (21). Because of our choice of $\mathbf{n}$, the normal derivatives are directed into the region A.

Now, from (19), since[9]

$$H_0^{(1)}(k_t R) = \frac{2i}{\pi} \log(k_t R) + 0(1), \quad \text{for} \quad k_t R \ll 1, \tag{23}$$

it follows that

$$\lim_{\epsilon \to 0} \int_{|\rho - \mathbf{r}| = \epsilon} \left(\psi^s \frac{\partial G}{\partial n} - G \frac{\partial \psi^s}{\partial n} \right) ds = -\psi^s(\mathbf{r}). \tag{24}$$

Also, since[11]

$$H_0^{(1)}(k_t R) \sim \left(\frac{2}{\pi k_t R} \right)^{1/2} \exp\left(i \left[k_t R - \frac{\pi}{4} \right] \right), \quad \text{for} \quad k_t R \gg 1, \tag{25}$$

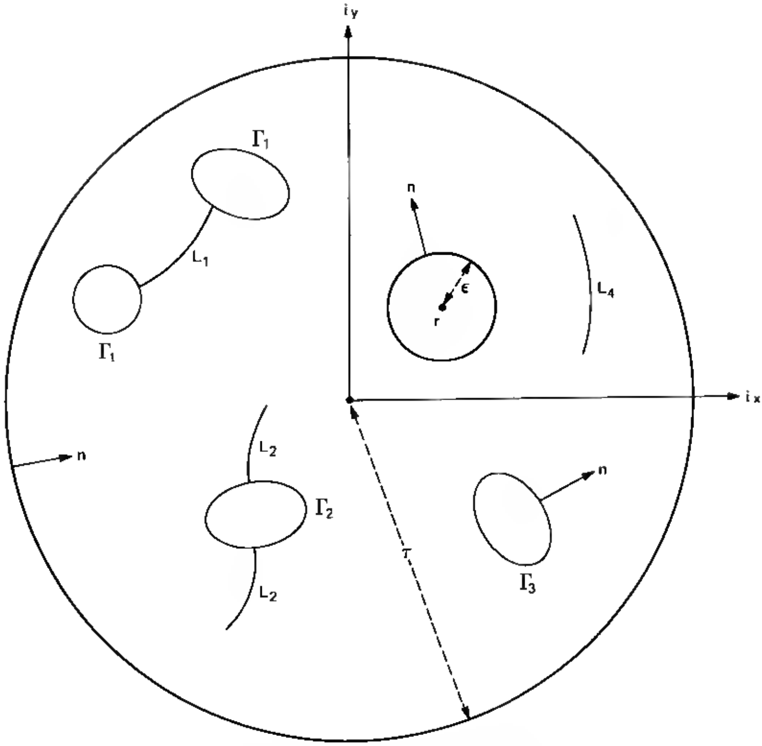

Fig. 1—Cross section of cylindrical scatterers.

$$\lim_{\tau \to \infty} \int_{|\rho|=\tau} \left(\psi^s \frac{\partial G}{\partial n} - G \frac{\partial \psi^s}{\partial n} \right) ds = 0, \tag{26}$$

if the scattered fields satisfy the radiation condition[12]

$$\lim_{\rho \to \infty} \rho^{1/2} \left(\frac{\partial \psi^s}{\partial \rho} - ik_t \psi^s \right) = 0, \quad (\rho = |\boldsymbol{\rho}|), \tag{27}$$

which we assume to be the case. If we let $\epsilon \to 0$ and $\tau \to \infty$ in (22), it follows from (24) and (26) that

$$\psi^s(\mathbf{r}) = \int_C \left[\psi^s(\boldsymbol{\rho}) \frac{\partial G}{\partial n} - G \frac{\partial \psi^s}{\partial n} (\boldsymbol{\rho}) \right] ds. \tag{28}$$

If we consider the incident field ψ^i, and apply Green's theorem in the region(s) enclosed by Γ_j, and use (16) and (21), we obtain

$$\int_{\Gamma_j} \left[\psi^i(\boldsymbol{\rho}) \frac{\partial G}{\partial n} - G \frac{\partial \psi^i}{\partial n} (\boldsymbol{\rho}) \right] ds = 0. \tag{29}$$

Also, because of the continuity of ψ^i and G, and their normal derivatives, on L_j, it follows that

$$\int_{L_j^+ \cup L_j^-} \left[\psi^i(\rho) \frac{\partial G}{\partial n} - G \frac{\partial \psi^i}{\partial n}(\rho) \right] ds = 0, \tag{30}$$

since the normals are reversed on opposite sides of L_j. Hence, since $C = \cup_j (\Gamma_j \cup L_j^+ \cup L_j^-)$,

$$\int_C \left[\psi^i(\rho) \frac{\partial G}{\partial n} - G \frac{\partial \psi^i}{\partial n}(\rho) \right] ds = 0. \tag{31}$$

Consequently, with the help of (17), we may rewrite (28) in the form

$$\psi^s(\mathbf{r}) = \int_C \left[\psi(\rho) \frac{\partial G}{\partial n} - G \frac{\partial \psi}{\partial n}(\rho) \right] ds. \tag{32}$$

The advantage of doing this is that, because of the boundary conditions in (18), one of the two terms in the integrand in (32) vanishes.

We first consider the case of E waves. Then, from (11), (19), and (32), we have

$$E_z^s(\mathbf{r}) = -\frac{i}{4} \int_C H_0^{(1)}(k_t R) \frac{\partial E_z}{\partial n}(\rho)\, ds. \tag{33}$$

If we now let $\mathbf{r}$ tend to a point on C, we obtain

$$E_z^i(\mathbf{r}) = \frac{i}{4} \int_C H_0^{(1)}(k_t R) \frac{\partial E_z}{\partial n}(\rho)\, ds, \quad \mathbf{r} \,\varepsilon\, C, \tag{34}$$

since $E_z^i + E_z^s = 0$ on C. We may rewrite (34) in the form

$$\begin{aligned} E_z^i(\mathbf{r}) &= \frac{i}{4} \int_\Gamma H_0^{(1)}(k_t R) \frac{\partial E_z}{\partial n}(\rho)\, ds \\ &+ \frac{i}{4} \int_L H_0^{(1)}(k_t R) \left\{ \left[\frac{\partial E_z}{\partial n}(\rho) \right]_+ + \left[\frac{\partial E_z}{\partial n}(\rho) \right]_- \right\} ds, \quad \mathbf{r} \,\varepsilon\, C, \end{aligned} \tag{35}$$

where $\Gamma = \cup_j \Gamma_j$ and $L = \cup_j L_j$. This integral equation may be used to determine the current density on Γ and the total current density on L, which suffices to determine the scattered field from (33). However, (35) does not yield the separate values of the normal derivative of E_z on either side of L. We will return to this point in the next section.

We now consider the case of H waves. Since[9]

$$\frac{d}{dR} H_0^{(1)}(k_t R) = -k_t H_1^{(1)}(k_t R), \tag{36}$$

it follows from (11), (19), (20), and (32) that

$$H_z^s(\mathbf{r}) = \frac{i}{4} k_t \int_C H_1^{(1)}(k_t R) \frac{\mathbf{R} \cdot \mathbf{n}}{R} H_z(\boldsymbol{\rho})\, ds. \tag{37}$$

Let $\mathbf{r}_0$ be a point on Γ, not at a corner, and let σ be a small segment of Γ containing $\mathbf{r}_0$. Then, as seen from Fig. 2, by considering the angle $\delta\phi$ subtended by tbe element δs of σ, and letting $\delta s \to 0$, we obtain

$$\frac{d\phi}{ds} = \frac{\mathbf{R} \cdot \mathbf{n}}{R^2}. \tag{38}$$

Since, from (23) and (36), $k_t R H_1^{(1)}(k_t R) \to -2i/\pi$ as $k_t R \to 0$, it follows that

$$\lim_{\mathbf{r} \to \mathbf{r}_0} \frac{i}{4} k_t \int_\sigma H_1^{(1)}(k_t R) \frac{\mathbf{R} \cdot \mathbf{n}}{R} H_z(\boldsymbol{\rho})\, ds \sim \frac{H_z(\mathbf{r}_0)}{2\pi} \int_\sigma d\phi. \tag{39}$$

But since we have assumed that $\mathbf{r}_0$ is not at a corner, the angle subtended at $\mathbf{r}_0$ by σ tends to π as the length of σ tends to zero. Hence, from (37), we have

$$H_z^s(\mathbf{r}) = \frac{1}{2} H_z(\mathbf{r}) + \frac{i}{4} k_t P \int_C H_1^{(1)}(k_t R) \frac{\mathbf{R} \cdot \mathbf{n}}{R} H_z(\boldsymbol{\rho})\, ds, \quad \mathbf{r} \in \Gamma', \tag{40}$$

where P denotes the principal value of the integral, corresponding to the limit of the integral over $C - \sigma$ as the length of σ tends to zero, and Γ' denotes Γ less its corners. Since $H_z = H_z^i + H_z^s$, we may rewrite (40) in the form

$$\begin{aligned} \frac{1}{2} H_z(\mathbf{r}) = H_z^i(\mathbf{r}) &+ \frac{i}{4} k_t P \int_\Gamma H_1^{(1)}(k_t R) \frac{\mathbf{R} \cdot \mathbf{n}}{n} H_z(\rho)\, ds \\ &+ \frac{i}{4} k_t \int_L H_1^{(1)}(k_t R) \frac{\mathbf{R}}{R} \cdot \{\mathbf{n}_+ [H_z(\boldsymbol{\rho})]_+ \\ &+ \mathbf{n}_- [H_z(\boldsymbol{\rho})]_-\}\, ds, \quad \mathbf{r} \in \Gamma'. \end{aligned} \tag{41}$$

Now let $\mathbf{r}_0$ be a point on L_j, not at a corner (or endpoint), and let σ be a small segment of L_j containing $\mathbf{r}_0$. Then, from (38), if we let $\mathbf{r}$ tend to $\mathbf{r}_0$ from the L_j^+ side, we obtain

$$\begin{aligned} \lim_{\mathbf{r} \to [\mathbf{r}_0]_+} \frac{i}{4} k_t \int_\sigma H_1^{(1)}(k_t R) \frac{\mathbf{R}}{R} \cdot \{\mathbf{n}_+ [H_z(\boldsymbol{\rho})]_+ + \mathbf{n}_- [H_z(\boldsymbol{\rho})]_-\}\, ds \\ \sim \frac{1}{2\pi} \{[H_z(\mathbf{r}_0)]_+ - [H_z(\mathbf{r}_0)]_-\} \int_\sigma d\phi, \end{aligned} \tag{42}$$

since $\mathbf{n}_- = -\mathbf{n}_+$. Hence from (37), it follows that

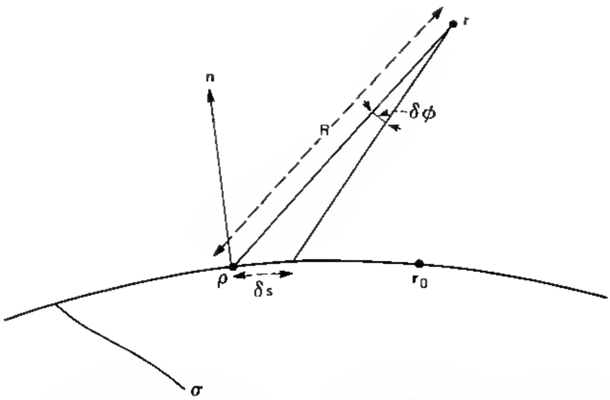

Fig. 2—Angle subtended by an element of a cross-sectional boundary curve.

$$
\begin{aligned}
&\frac{i}{4} k_t \int_\Gamma H_1^{(1)}(k_t R) \frac{\mathbf{R}\cdot\mathbf{n}}{R} H_z(\rho)\, ds \\
&+ \frac{i}{4} k_t P \int_L H_1^{(1)}(k_t R) \frac{\mathbf{R}}{R} \cdot \{\mathbf{n}_+ [H_z(\rho)]_+ + \mathbf{n}_- [H_z(\rho)]_-\}\, ds \\
&= [H_z^s(\mathbf{r})]_+ - \frac{1}{2}\{[H_z(\mathbf{r})]_+ - [H_z(\mathbf{r})]_-\} \\
&= \frac{1}{2}\{[H_z(\mathbf{r})]_+ + [H_z(\mathbf{r})]_-\} - H_z^i(\mathbf{r}), \quad \mathbf{r} \in L',
\end{aligned} \tag{43}
$$

where L' denotes L less its corners (and endpoints). The same result is obtained by letting $\mathbf{r}$ tend to $\mathbf{r}_0$ from the L_j^- side, as is evident from the symmetry in (43). We have made use of the continuity of $H_z^i(\mathbf{r})$.

Now $\mathbf{n}_+ = -\mathbf{n}_-$ on L, but we note that the integral equations (41) and (43) do not determine $H_z(\mathbf{r})$ for $\mathbf{r} \in \Gamma'$, and $\{[H_z(\mathbf{r})]_+ - [H_z(\mathbf{r})]_-\}$ for $\mathbf{r} \in L'$, because of the unknown quantity $[H_z(\mathbf{r})]_+ + [H_z(\mathbf{r})]_-$ on the right-hand side of (43). Moreover, if L consists of segments of a straight line, then $\mathbf{R}\cdot\mathbf{n} = 0$ for $\mathbf{r} \in L'$, $\rho \in L'$ and $\rho \neq \mathbf{r}$, and the second integral in (43) vanishes. If, in addition, Γ is empty, then (43) reduces to

$$
[H_z(\mathbf{r})]_+ + [H_z(\mathbf{r})]_- = 2H_z^i(\mathbf{r}), \quad \mathbf{r} \in L'. \tag{44}
$$

This reduction was pointed out by Noble[3] in the case of an infinitesimally thin strip and by Millar[13] in the case of coplanar strips. In the next section, we derive an integral equation which does not degenerate, in the case of H waves, for $\mathbf{r} \in L'$. We remark that the integral equation (35) does not degenerate for $\mathbf{r} \in L$, and this is presumably because it was obtained by setting $E_z^s = -E_z^i$ on C. This suggests that we should derive an expression for $\partial H_z^s/\partial n$ on C', and set it equal to $-\partial H_z^i/\partial n$.

IV. INTEGRAL EQUATIONS DERIVED FROM THE GRADIENT OF THE SCATTERED FIELD

We now return to the integral representation (32) for the scattered field, and calculate the transverse component of its gradient. If we substitute the explicit form (19) of the Green's function into (32), and use (20) and (36), we obtain

$$\psi^s(\mathbf{r}) = \frac{i}{4}\int_C \left[k_t H_1^{(1)}(k_k R)\frac{\mathbf{R}\cdot\mathbf{n}}{R}\psi(\boldsymbol{\rho}) - H_0^{(1)}(k_t R)\frac{\partial\psi}{\partial n}(\boldsymbol{\rho})\right] ds. \quad (45)$$

Hence, since[9]

$$\frac{d}{dR}[RH_1^{(1)}(k_t R)] = k_t R H_0^{(1)}(k_t R), \quad (46)$$

it follows that

$$\nabla_t\psi^s(\mathbf{r}) = \frac{i}{4}k_t^2 \int_C H_0^{(1)}(k_t R)\frac{(\mathbf{R}\cdot\mathbf{n})\mathbf{R}}{\mathrm{R}^2}\psi(\boldsymbol{\rho})\,ds$$

$$+\frac{i}{4}k_t\int_C H_1^{(1)}(k_t R)\left[\frac{\mathbf{n}}{R} - \frac{2(\mathbf{R}\cdot\mathbf{n})\mathbf{R}}{R^3}\right]\psi(\boldsymbol{\rho})\,ds$$

$$+\frac{i}{4}k_t\int_C H_1^{(1)}(k_t R)\frac{\mathbf{R}}{R}\frac{\partial\psi}{\partial n}(\boldsymbol{\rho})\,ds. \quad (47)$$

Now, since $\partial\boldsymbol{\rho}/\partial s = \mathbf{t}$, and $\mathbf{t}\times\mathbf{i}_z = -\mathbf{n}$, we have

$$\frac{\partial}{\partial s}\left[\frac{\mathbf{R}\times\mathbf{i}_z}{R^2}\right] = \frac{\mathbf{n}}{R^2} + \frac{2}{R^4}(\mathbf{R}\times\mathbf{i}_z)(\mathbf{R}\cdot\mathbf{t}). \quad (48)$$

Also,

$$\begin{aligned}(\mathbf{R}\times\mathbf{i}_z)(\mathbf{R}\cdot\mathbf{t}) &= (\mathbf{R}\cdot\mathbf{t})[(\mathbf{R}\cdot\mathbf{n})\mathbf{t} - (\mathbf{R}\cdot\mathbf{t})\mathbf{n}]\\ &= (\mathbf{R}\cdot\mathbf{n})\mathbf{R} - \mathrm{R}^2\mathbf{n}. \end{aligned} \quad (49)$$

Hence,

$$\frac{\mathbf{n}}{R^2} - \frac{2(\mathbf{R}\cdot\mathbf{n})\mathbf{R}}{R^4} = -\frac{\partial}{\partial s}\left(\frac{\mathbf{R}\times\mathbf{i}_z}{R^2}\right). \quad (50)$$

If we substitute (50) into the second integral in (47), and integrate by parts, and combine terms with the help of (49), we obtain

$$\nabla_t\psi^s(\mathbf{r}) = \frac{i}{4}k_t^2\int_C H_0^{(1)}(k_t R)\mathbf{n}\psi(\boldsymbol{\rho})\,ds$$

$$+\frac{i}{4}k_t\int_C \frac{H_1^{(1)}(k_tR)}{R}\left[(\mathbf{R}\times \mathbf{i}_z)\frac{\partial\psi}{\partial s}(\rho)+\mathbf{R}\frac{\partial\psi}{\partial n}(\rho)\right]ds. \tag{51}$$

This expression for the gradient of the scattered field is the 2-dimensional analog of that derived by Mitzner[4] in 3 dimensions. We give an alternate derivation of (51) in the appendix.

We are interested in calculating the normal derivative of the scattered field in the limit as $\mathbf{r}$ tends to a point $\mathbf{r}_0 \in \Gamma' \cup L'$, i.e. $\nabla_t\psi^s(\mathbf{r})\cdot\mathbf{n}_0$, where $\mathbf{n}_0$ is a unit normal to $\Gamma' \cup L'$ at $\mathbf{r}_0$. It will be seen that the limiting value of this quantity may be calculated with the help of (51), whereas the second integral in (47) has a singular behavior. However, it is not necessary to integrate this second integral by parts completely around C, as was done to obtain (51). If we let $\sum_0$ be a segment (or segments) of $\Gamma \cup L$ which has $\mathbf{r}_0$ as an interior point, then it suffices to integrate by parts over $\sum_0$. Since the second integral in (47) vanishes in the case of E waves, because $E_z = 0$ on the boundary, we now consider the case of H waves.

We define

$$\mathbf{n}_0 = \begin{array}{l}\mathbf{n}(\mathbf{r}_0), \quad \mathbf{r}_0\in\Gamma',\\ \mathbf{n}_+(\mathbf{r}_0) = -\mathbf{n}_-(\mathbf{r}_0), \quad \mathbf{r}_0 \in L',\end{array} \tag{52}$$

and choose

$$\mathbf{n}=\mathbf{n}_+ \Rightarrow d\rho/ds = \mathbf{t} = \mathbf{n}_+\times\mathbf{i}_z, \quad \text{on } L. \tag{53}$$

We also define the tangential component of current density

$$J(\rho) = \begin{array}{l} H_z(\rho), \quad \rho\in\Gamma,\\ [H_z(\rho)]_+ - [H_z(\rho)]_-, \quad \rho\in L.\end{array} \tag{54}$$

Then, from (11) and (47), after an integration by parts, and use of (46), (49), and (50), we obtain

$$\nabla_t H_z^s(\mathbf{r})\cdot\mathbf{n}_0 = \frac{i}{4}k_t^2\int_{\Gamma\cup L-\Sigma_0} H_0^{(1)}(k_tR)\frac{(\mathbf{R}\cdot\mathbf{n})(\mathbf{R}\cdot\mathbf{n}_0)}{R^2}J(\rho)\,ds$$

$$+\frac{i}{4}k_t\int_{\Gamma\cup L-\Sigma_0} H_1^{(1)}(k_tR)\left[\frac{\mathbf{n}}{R}-\frac{2(\mathbf{R}\cdot\mathbf{n})\mathbf{R}}{R^3}\right]\cdot\mathbf{n}_0 J(\rho)\,ds$$

$$+\frac{i}{4}k_t^2\int_{\Sigma_0} H_0^{(1)}(k_tR)\mathbf{n}\cdot\mathbf{n}_0 J(\rho)\,ds$$

$$+\frac{i}{4}k_t\int_{\Sigma_0} H_1^{(1)}(k_tR)\frac{(\mathbf{R}\times\mathbf{i}_z)}{R}\cdot\mathbf{n}_0\frac{\partial J}{\partial s}(\rho)\,ds$$

$$-\frac{i}{4}k_t\left[H_1^{(1)}(k_tR)\frac{(\mathbf{R}\times\mathbf{i}_z)}{R}\cdot\mathbf{n}_0J(\rho)\right]_{\Sigma_0} \tag{55}$$

The contributions from all the endpoints of $\sum_0$ must be included in the last term in (55). If $\sum_0 = \Gamma \cup L$, then this last term is zero, and the first two integrals in (55) are absent.

We now consider $\mathbf{r} \to \mathbf{r}_0$, in a direction which is not tangential to $\Gamma' \cup L'$. Since $\mathbf{r}_0$ is an interior point of $\sum_0$, the first two integrals in (55) are well-behaved as $\mathbf{r} \to \mathbf{r}_0$, as are the contributions from the endpoints of $\sum_0$, represented by the last term in (55). Also, it follows from (23) that, for $\mathbf{r} = \mathbf{r}_0$, there is an integrable singularity in the third integral in (55). It remains to consider the fourth integral in (55). As depicted in Fig. 3, we take

$$\mathbf{r} = \mathbf{r}_0 + \epsilon(\cos\chi\mathbf{t}_0 + \sin\chi\mathbf{n}_0), \tag{56}$$

where $\sin\chi \neq 0$. Also, for convenience, we take $s = 0$ at $\mathbf{r} = \mathbf{r}_0$, so that[14]

$$\rho = \mathbf{r}_0 + s\mathbf{t}_0 - \frac{1}{2}\kappa_0 s^2\mathbf{n}_0 + 0(s^3) \tag{57}$$

for small $|s|$, where κ_0 is the curvature at $\mathbf{r}_0$. Hence,

$$\mathbf{R} = \mathbf{r} - \rho = (\epsilon\cos\chi - s)\mathbf{t}_0 + (\epsilon\sin\chi + \frac{1}{2}\kappa_0 s^2)\mathbf{n}_0 + 0(s^3), \tag{58}$$

and

$$R^2 = |\mathbf{R}|^2 = (s - \epsilon\cos\chi)^2 + \epsilon^2\sin^2\chi + 0(\epsilon s^2) + 0(s^4). \tag{59}$$

From (52), (53), and (58), it follows that

$$(\mathbf{R}\times\mathbf{i}_z)\cdot\mathbf{n}_0 = -\mathbf{R}\cdot\mathbf{t}_0 = (s - \epsilon\cos\chi) + 0(s^3). \tag{60}$$

But[9]

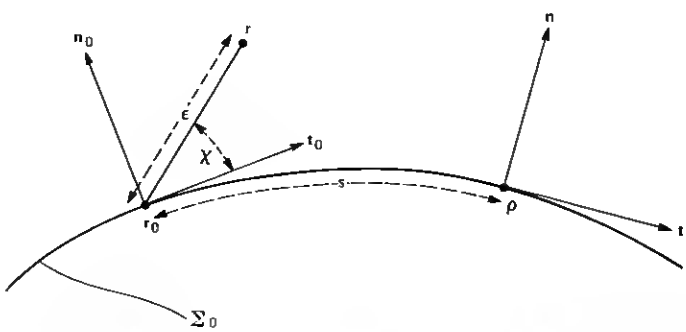

Fig. 3—Coordinates of a point in the neighborhood of a cross-sectional boundary curve.

$$H_1^{(1)}(k_tR) = \frac{-2i}{\pi k_tR} + 0[k_tR\log(k_tR)], \quad \text{for} \quad k_tR \ll 1, \tag{61}$$

and

$$\int_{-\delta}^{\delta} \frac{(s - \epsilon\cos\chi)\,ds}{[(s-\epsilon\cos\chi)^2 + \epsilon^2\sin^2\chi]}$$

$$= \frac{1}{2}\log\left[\frac{(\delta-\epsilon\cos\chi)^2+\epsilon^2\sin^2\chi}{(\delta+\epsilon\cos\chi)^2+\epsilon^2\sin^2\chi}\right]. \tag{62}$$

It follows from (59) to (62) that

$$\lim_{\delta\to 0}\left\{\lim_{\epsilon\to 0}\int_{-\delta}^{\delta} H_1^{(1)}(k_tR)\frac{(\mathbf{R}\times\mathbf{i}_z)}{R}\cdot\mathbf{n}_0\frac{\partial J}{\partial s}(\rho)\,ds\right\} = 0. \tag{63}$$

Hence the principal value of the fourth integral in (55) must be taken in the limit $\mathbf{r}\to\mathbf{r}_0$.

Having shown that the right-hand side of (55) is meaningful in the limit $\mathbf{r}\to\mathbf{r}_0$, we now note that the left-hand side tends to $\partial H_z^s/\partial n_0 = -\partial H_z^i/\partial n_0$, since $\partial H_z/\partial n = 0$ on the boundary, and hence its value is known. Hence the limit of (55) as $\mathbf{r}\to\mathbf{r}_0\in\Gamma'\cup L'$ leads to the desired integral equation for $J(\rho)$, as defined in (54). We remark that $\partial J/\partial s$, as well as J, occurs in the integrand. We also remark that, when this integral equation has been solved for $J(\mathbf{r})$ for $\mathbf{r}\in\Gamma'\cup L'$, then (43) may be used to calculate $[H_z(\mathbf{r})]_+ + [H_z(\mathbf{r})]_-$ for $\mathbf{r}\in L'$, and hence the separate values of $[H_z(\mathbf{r})]_+$ and $[H_z(\mathbf{r})]_-$. We comment that we could presumably use the integral equation derived from (55) for $\mathbf{r}_0\in L'$ only, and combine it with (41) for $\mathbf{r}\in\Gamma'$, to solve for $J(\mathbf{r})$ for $\mathbf{r}\in\Gamma'\cup L'$.

We now consider the case of E waves. Then, from (11) and (47), or equivalently (51),

$$\nabla_t E_z^s(\mathbf{r})\cdot\mathbf{n}_0 = \frac{i}{4}k_t\int_\Gamma H_1^{(1)}(k_tR)\frac{\mathbf{R}\cdot\mathbf{n}_0}{R}\frac{\partial E_z}{\partial n}(\rho)\,ds$$

$$+\frac{i}{4}k_t\int_L H_1^{(1)}(k_tR)\frac{\mathbf{R}\cdot\mathbf{n}_0}{R}\left\{\left[\frac{\partial E_z}{\partial n}(\rho)\right]_+ + \left[\frac{\partial E_z}{\partial n}(\rho)\right]_-\right\}ds. \tag{64}$$

But, from (58),

$$\mathbf{R}\cdot\mathbf{n}_0 = \epsilon\sin\chi + 0(s^2), \tag{65}$$

and, for $\delta > 0$,

$$\lim_{\epsilon\to 0+}\int_{-\delta}^{\delta}\frac{\epsilon\sin\chi\,ds}{[(s-\epsilon\cos\chi)^2+\epsilon^2\sin^2\chi]}$$

$$=\lim_{\epsilon\to 0+}\left[\tan^{-1}\left(\frac{s-\epsilon\cos\chi}{\epsilon\sin\chi}\right)\right]_{-\delta}^{\delta}=\pi\,\mathrm{sgn}(\sin\chi). \quad (66)$$

We first consider $\mathbf{r}\to\mathbf{r}_0\in\Gamma'$. Then, from (52) and Fig. 3, $\sin\chi>0$. Hence, from (64), with the help of (59), (61), (65), and (66), we obtain

$$\frac{i}{4}k_tP\int_\Gamma H_1^{(1)}(k_tR_0)\frac{\mathbf{R}_0\cdot\mathbf{n}_0}{R_0}\frac{\partial E_z}{\partial n}(\rho)\,ds$$

$$+\frac{i}{4}k_t\int_L H_1^{(1)}(k_tR_0)\frac{\mathbf{R}_0\cdot\mathbf{n}_0}{R_0}\left\{\left[\frac{\partial E_z}{\partial n}(\rho)\right]_++\left[\frac{\partial E_z}{\partial n}(\rho)\right]_-\right\}ds$$

$$=\frac{\partial E_z^s}{\partial n}(\mathbf{r}_0)-\frac{1}{2}\frac{\partial E_z}{\partial n}(\mathbf{r}_0)=\frac{1}{2}\frac{\partial E_z}{\partial n}(\mathbf{r}_0)-\frac{\partial E_z^i}{\partial n}(\mathbf{r}_0),\quad \mathbf{r}_0\in\Gamma', \quad (67)$$

where $\mathbf{R}_0=\mathbf{r}_0-\rho$.

We now consider $\mathbf{r}\to\mathbf{r}_0\in L'$. If the approach is from the plus side then, from (52) and Fig. 3, $\mathbf{n}_0=\mathbf{n}_+(\mathbf{r}_0)$ and $\sin\chi>0$. Hence, from (64), with the help of (59), (61), (65), and (66), it follows that

$$\frac{i}{4}k_t\int_\Gamma H_1^{(1)}(k_tR_0)\frac{\mathbf{R}_0\cdot\mathbf{n}_0}{R_0}\frac{\partial E_z}{\partial n}(\rho)\,ds$$

$$+\frac{i}{4}k_tP\int_L H_1^{(1)}(k_tR_0)\frac{\mathbf{R}_0\cdot\mathbf{n}_0}{R_0}\left\{\left[\frac{\partial E_z}{\partial n}(\rho)\right]_++\left[\frac{\partial E_z}{\partial n}(\rho)\right]_-\right\}ds$$

$$=\left[\frac{\partial E_z^s}{\partial n}(\mathbf{r}_0)\right]_+-\frac{1}{2}\left\{\left[\frac{\partial E_z}{\partial n}(\mathbf{r}_0)\right]_++\left[\frac{\partial E_z}{\partial n}(\mathbf{r}_0)\right]_-\right\}$$

$$=\frac{1}{2}\left\{\left[\frac{\partial E_z}{\partial n}(\mathbf{r}_0)\right]_+-\left[\frac{\partial E_z}{\partial n}(\mathbf{r}_0)\right]_-\right\}-\left[\frac{\partial E_z^i}{\partial n}(\mathbf{r}_0)\right]_+,\quad \mathbf{r}_0\in L'. \quad (68)$$

On the other hand, if the approach to $\mathbf{r}_0$ is from the minus side, then $\mathbf{n}_0=-\mathbf{n}_-(\mathbf{r}_0)$ and $\sin\chi<0$, and it follows that the left-hand side of (68) is equal to

$$-\left[\frac{\partial E_z^s}{\partial n}(\mathbf{r}_0)\right]_-+\frac{1}{2}\left\{\left[\frac{\partial E_z}{\partial n}(\mathbf{r}_0)\right]_++\left[\frac{\partial E_z}{\partial n}(\mathbf{r}_0)\right]_-\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{\partial E_z}{\partial n}(\mathbf{r}_0)\right]_+ - \left[\frac{\partial E_z}{\partial n}(\mathbf{r}_0)\right]_-\right\} + \left[\frac{\partial E_z^i}{\partial n}(\mathbf{r}_0)\right]_- . \qquad (69)$$

Since $[\partial E_z^i/\partial n\ (\mathbf{r}_0)]_- = -[\partial E_z^i/\partial n\ (\mathbf{r}_0)]_+$, for $\mathbf{r}_0 \in L'$, we again obtain (68).

We remark that when the integral equation (35) has been used to determine $\partial E_z/\partial n\ (\mathbf{r})$ for $\mathbf{r} \in \Gamma'$ and $[\partial E_z/\partial n\ (\mathbf{r})]_+ + [\partial E_z/\partial n\ (\mathbf{r})]_-$ for $\mathbf{r} \in L'$, then (68) may be used to calculate $[\partial E_z/\partial n\ (\mathbf{r})]_+ - [\partial E_z/\partial n\ (\mathbf{r})]_-$ for $\mathbf{r} \in L'$, and hence the separate values of $[\partial E_z/\partial n\ (\mathbf{r})]_+$ and $[\partial E_z/\partial n\ (\mathbf{r})]_-$. This is analogous to tbe earlier remark concerning the use of (43). If L consists of segments of a straight line, then $\mathbf{R}_0 \cdot \mathbf{n}_0 = 0$ for $\mathbf{r}_0 \in L'$, $\boldsymbol{\rho} \in L'$ and $\boldsymbol{\rho} \neq \mathbf{r}_0$, and the second integral in (68) vanishes. If, in addition, Γ is empty, then (68) reduces to

$$\left[\frac{\partial E_z}{\partial n}(\mathbf{r}_0)\right]_+ - \left[\frac{\partial E_z}{\partial n}(\mathbf{r}_0)\right]_- = 2\left[\frac{\partial E_z^i}{\partial n}(\mathbf{r}_0)\right]_+, \quad \mathbf{r}_0 \in L'. \qquad (70)$$

This reduction was pointed out by Noble[3] in the case of an infinitesimally thin strip.

It is of interest to note that (35), and the integral equation obtained from (55) by letting $\mathbf{r} \to \mathbf{r}_0 \in \Gamma' \cup L'$, are Fredholm equations of the first kind. On the other hand, (41) and (67) which hold for $\mathbf{r} \in \Gamma'$, and Fredholm equations of the second kind, which is usually preferable from the viewpoint of numerical solution. It is somewhat unfortunate, in this respect, that the corresponding equations which hold for $\mathbf{r} \in L'$, namely (43) and (68), do not determine $[\mathrm{H}_z(\mathbf{r})]_+ - [H_z(\mathbf{r})]_-$ and $[\partial \mathrm{E}_z/\partial n\ (\mathbf{r})]_+ + [\partial \mathrm{E}_z/\partial n\ (\mathbf{r})]_-$.

NOTE ADDED IN PROOF

In the 3-dimensional case of an open thin shell, the use of a degenerate integral equation, analogous to (68), to determine the current densities on both sides of the shell, once the total current density is known, was pointed out by Stakgold.[16]

APPENDIX

We here give an alternate derivation of the integral equation (51). Let $\mathbf{e}$ be a constant vector. Then from (16), it follows that

$$(\nabla_t^2 + k_t^2)(\mathbf{e}\cdot\nabla_t\psi^s) = 0, \quad (\nabla_t^2 + k_t^2)(\mathbf{e}\cdot\nabla_t\psi^i) = 0. \qquad (71)$$

If we apply Green's theorem, as in Section III, but this time to $\mathbf{e}\cdot\nabla_t\psi^s$ and G, then, with the help of (21) and (27), we obtain

$$\mathbf{e}\cdot\nabla_t\psi^s(\mathbf{r}) = \int_C \left\{[\mathbf{e}\cdot\nabla_t'\psi^s(\boldsymbol{\rho})]\frac{\partial G}{\partial n} - G\frac{\partial}{\partial n}[\mathbf{e}\cdot\nabla_t'\psi^s(\boldsymbol{\rho})]\right\} ds, \qquad (72)$$

where ∇'_t denotes the transverse component of the gradient with respect to the coordinates of $\boldsymbol{\rho}$. In a manner analogous to that used in Section III, we also obtain

$$\int_C \left\{ [\mathbf{e}\cdot\nabla'_t\psi^i(\boldsymbol{\rho})] \frac{\partial G}{\partial n} - G \frac{\partial}{\partial n} [\mathbf{e}\cdot\nabla'_t\psi^i(\boldsymbol{\rho})] \right\} ds = 0. \tag{73}$$

Hence, with the help of (17), we may rewrite (72) in the form

$$\mathbf{e}\cdot\nabla_t\psi^s(\mathbf{r}) = \int_C \left\{ [\mathbf{e}\cdot\nabla'_t\psi(\boldsymbol{\rho})] \frac{\partial G}{\partial n} - G \frac{\partial}{\partial n} [\mathbf{e}\cdot\nabla'_t\psi(\boldsymbol{\rho})] \right\} ds. \tag{74}$$

Now

$$\frac{\partial}{\partial n} (\mathbf{e}\cdot\nabla'_t\psi) = \mathbf{n}\cdot\nabla'_t(\mathbf{e}\cdot\nabla'_t\psi), \tag{75}$$

and[7]

$$\begin{aligned}\nabla'_t(\mathbf{e}\cdot\nabla'_t\psi) = (\mathbf{e}\cdot\nabla'_t)\,(\nabla'_t\psi) &= \mathbf{e}\nabla'^2_t\psi - \nabla'_t \times (\mathbf{e} \times \nabla'_t\psi) \\ &= -\,\mathbf{e}k_t^2\psi - \nabla'_t \times (\mathbf{e} \times \nabla'_t\psi),\end{aligned} \tag{76}$$

from (16). But[7]

$$\nabla'_t \times [G(\mathbf{e} \times \nabla'_t\psi)] = \nabla'_t G \times (\mathbf{e} \times \nabla'_t\psi) + G\nabla'_t \times (\mathbf{e} \times \nabla'_t\psi), \tag{77}$$

and, from Stokes' theorem,[15]

$$\int_C \mathbf{n}\cdot\{\nabla'_t \times [G(\mathbf{e} \times \nabla'_t\psi)]\}\, ds = 0. \tag{78}$$

Hence,

$$\begin{aligned}\int_C G\mathbf{n}\cdot[\nabla'_t \times (\mathbf{e} \times \nabla'_t\psi)]\,ds &= -\int_C \mathbf{n}\cdot[\nabla'_t G \times (\mathbf{e} \times \nabla'_t\psi)]\,ds \\ = -\int_C (\mathbf{n} \times \nabla'_t G)\cdot\,(\mathbf{e} \times \nabla'_t\psi)\,ds &= \int_C \mathbf{e}\cdot[(\mathbf{n} \times \nabla'_t G) \times \nabla'_t\psi]\,ds.\end{aligned} \tag{79}$$

It follows, from (74) to (76) and (79), that

$$\mathbf{e}\cdot\left\{ \nabla_t\psi^s(\mathbf{r}) - \int_C \left[\frac{\partial G}{\partial n} \nabla'_t\psi + k_t^2 G\mathbf{n}\psi(\boldsymbol{\rho}) + (\mathbf{n} \times \nabla'_t G) \times \nabla'_t\psi \right] ds \right\} = 0. \tag{80}$$

Since $\mathbf{e}$ is an arbitrary (constant) vector, the expression in the curly brackets in (80) must vanish. But

$$(\nabla'_t G\cdot\mathbf{n})\nabla'_t\psi + (\mathbf{n} \times \nabla'_t G) \times \nabla'_t\psi = (\nabla'_t\psi\cdot n)\nabla'_t G + (\mathbf{n} \times \nabla'_t\psi) \times \nabla'_t G, \tag{81}$$

and $\mathbf{n} \times \nabla_t'\psi = -\mathbf{i}_z \, \partial\psi/\partial s$ on C. Hence, we obtain

$$\nabla_t \psi^s(\mathbf{r}) = \int_C \left[k_t^2 \, G\mathbf{n}\psi(\rho) + (\nabla_t' G) \frac{\partial \psi}{\partial n} + (\nabla_t' G \times \mathbf{i}_z) \frac{\partial \psi}{\partial s} \right] ds. \quad (82)$$

If we substitute for G from (19) and make use of (20) and (36), we obtain (51). The representation (82) for the gradient of the scattered field is the 2-dimensional analog of that derived by Mitzner[4] in 3 dimensions. We remark that the above derivation differs from his in that we used Green's second identity, whereas he used Green's first identity, and consequently some different transformations to reduce the result to the form corresponding to (82). Mitzner derived his result for the field inside a bounded volume, whereas we are considering scattered fields outside bounded cross-sectional curves, and have used Green's second identity so that we could make use of the radiation condition.

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